Topological aspects of poset spaces

Carl Mummert*

and Frank Stephan†

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Abstract

We study two classes of spaces whose points are filters on partially ordered sets. Points in MF spaces are maximal filters, while points in UF spaces are unbounded filters. We give a thorough account of the topological properties of these spaces. We obtain a complete characterization of the class of countably based MF spaces: they are precisely the second-countable T_1 spaces with the strong Choquet property. We apply this characterization to domain theory to characterize the class of second-countable spaces with a domain representation.

1 Introduction

Recent work in mathematical logic [10, 11, 12] has led to an interest in certain topological spaces formed from filters on partially ordered sets. This paper describes the general topology of these poset spaces.

*`mummertc@marshall.edu`. C. Mummert was partially supported by a VIGRE graduate traineeship under NSF Grant DMS-9810759 at the Pennsylvania State University.

†`fstephan@comp.nus.edu.sg`. F. Stephan is supported in part by NUS grant number R252-000-212-112.

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The results of the paper are divided as follows. In Section 2, we define two classes of spaces, MF spaces and UF spaces. Together these spaces form the class of poset spaces. We show that many familiar spaces are homeomorphic to poset spaces. In Section 3, we characterize the separation properties of poset spaces and show that any second-countable poset space is homeomorphic to a space of the same kind formed from a countable poset. In Section 4, we show that the class of MF spaces are closed under arbitrary topological products and that any G_δ subspace of an MF space is again an MF space. We show that UF spaces are closed under taking G_δ subspaces but not closed under binary products. In Section 5, we establish that poset spaces are of the second Baire category and possess the strong Choquet property. We give a characterization of the class of countably based MF spaces as the class of second-countable T_1 spaces with the strong Choquet property. In Section 6, we apply the results of Section 5 to domain theory, giving a complete characterization of the second-countable topological spaces that have a domain representation. Section 7 contains results on the relationship between MF spaces (not necessarily countably based) and semi-topogenous orders. We use semi-topogenous orders to establish a sufficient condition for an arbitrary space to be homeomorphic to an MF space. In Section 8, we show that every second-countable poset space is either countable or contains a perfect closed set.

2 Poset spaces

Our goal in this section is to define the class of poset spaces and show that this class includes all complete metric spaces and all locally compact Hausdorff spaces. We first review some basic definitions about partially ordered sets.

A *poset* is a set P with an reflexive, antisymmetric, transitive relation $\preceq$. That is, the following conditions hold for all p, q and r in P.

(1) $p \preceq p$.

(2) If $p \preceq q$ and $q \preceq p$ then $q = p$.

(3) If $p \preceq q$ and $q \preceq r$ then $p \preceq r$.

We write $p \prec q$ if $p \preceq q$ and $p \neq q$. If there is no r such that $r \preceq p$ and $r \preceq q$ then we write $p \perp q$.

A *filter* is a subset F of a poset P satisfying the following two conditions.

(1) For every $p, q \in F$ there is an $r \in F$ such that $r \preceq p$ and $r \preceq q$.

(2) For every $p \in F$ and $q \in P$ if $p \preceq q$ then $q \in F$.

A filter F is *unbounded* if there is no $r \in P$ such that $r \prec q$ for every $q \in F$. Furthermore, F is *maximal* if there is no strictly larger filter containing F. Every maximal filter is unbounded, but in general not every unbounded filter is maximal.

For any poset P, we let $\mathrm{UF}(P)$ denote the set of unbounded filters on P and let $\mathrm{MF}(P)$ denote the set of maximal filters on P. We topologize $\mathrm{UF}(P)$ with the basis $\{N_p \mid p \in P\}$, where

$$N_p = \{F \in \mathrm{UF}(P) \mid p \in F\}.$$

We give $\mathrm{MF}(P)$ the topology it inherits as a subset of $\mathrm{UF}(P)$; when we work with spaces of maximal filters we may write N_p to denote the set of maximal filters containing p. To facilitate the exposition, we sometimes identify $p \in P$ with the open set N_p and identify a subset U of P with the open set $\bigcup_{p \in U} N_p$.

A *UF space* is a space of the form $\mathrm{UF}(P)$ and an *MF space* is a space of the form $\mathrm{MF}(P)$. UF spaces and MF spaces are collectively referred to as *poset spaces*. A poset space is *countably based* if it is formed from a countable poset. It is possible that P is uncountable but $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$ is a second-countable space (an example is provided after Theorem 2.3). We will show below that every second-countable poset space is homeomorphic to a countably based poset space. This result justifies our terminology.

Remark 2.1. It is sometimes convenient to work with strict partial orders instead of the non-strict partial orders defined above. A strict partial order is a set P with an irreflexive, transitive relation $\prec$. Every strict partial order $\langle P, \prec \rangle$ is canonically associated to non-strict partial order $\langle P, \preceq \rangle$ in which $p \preceq q$ if and only if $p \prec q$ or $p = q$, and every non-strict partial order arises in this way. A filter on a strict partial order $\langle P, \prec \rangle$ is a set $F \subseteq P$ that is upward closed and such that if $p, q \in F$ then there is an $r \in F$ with $r \preceq p$ and $r \preceq q$.

It follows immediately from these definitions that if $\langle P, \prec \rangle$ is a strict partial order, $\langle P, \preceq \rangle$ is the corresponding non-strict partial order, and $F \subseteq P$, then F is a filter in $\langle P, \prec \rangle$ if and only if F is a filter in $\langle P, \preceq \rangle$, and *vice versa*. Moreover, F is a maximal (unbounded) filter in either of these partial orders if and only if it is maximal (unbounded, respectively) in the other partial order.

A topology on the set of maximal (unbounded) filters of a strict partial is defined in the same way as for a non-strict partial order. Once this definition is made, it is immediate that for any strict poset $\langle P, \prec \rangle$ and corresponding non-strict poset $\langle P, \preceq \rangle$, the identity map $P \to P$ induces a homeomorphism of the topological spaces of maximal (unbounded, respectively) filters of these posets. For this reason, when it is convenient, we may prove results using strict partial orders instead of non-strict partial orders. This technique is sound because any example of a poset space obtained from a strict partial order can be converted to a homeomorphic example obtained from a non-strict poset space, and *vice versa*.

We now present two examples showing that many familiar spaces are homeomorphic to poset spaces.

Theorem 2.2. Every locally compact Hausdorff space is homeomorphic to an MF space.

Proof. Let X be a locally compact Hausdorff space and let P be the set of all nonempty precompact open subsets of X. For U,$V \in P$ we put $U \preceq V$ if $U = V$ or the closure of U is contained in V. If F is a filter and $U \in F$ then, because U is precompact,

$$\bigcap F = \bigcap \{\bar{V} \mid \bar{V} \subseteq U, V \in F\}$$

is the filtered intersection of non-empty compact sets, and hence is non-empty and compact. Since X is Hausdorff, any two points of X have open neighborhoods whose closures are disjoint. If F is a a maximal filter, then at most one of these neighborhoods can be in F, which implies that $\bigcap F$ is a singleton. Finally, the mapping $\phi\colon \mathrm{MF}(P) \to X$ given by $F \mapsto \bigcap F$ has as its inverse the mapping

$$\phi^{-1}\colon x \mapsto \{p \in P \mid x \in N_p\}.$$

To prove that ϕ is continuous, fix $x \in \mathrm{MF}(P)$ and let U be any open neighborhood of $\phi(x)$ in X. Because X is locally compact, we may assume without loss of generality that U is precompact, because the precompact sets form a basis for the topology. Thus we assume $U = N_p$ for some $p \in P$. Now, since $\phi(x) \in U$, we have $p \in x$, so $x \in N_p$. Moreover, for any $F \in N_p$ in $\mathrm{MF}(P)$, we have $\phi(F) = \bigcap F \in U$. This shows ϕ is continuous.

To prove that ϕ^{-1} is continuous, let $y \in X$ be fixed, and let V be any open neighbohood of $\phi^{-1}(y)$ in $\mathrm{MF}(P)$. Without loss of generality we may assume $V = N_p$ for some $p \in P$. Now p itself is some precompact open subset U of X, and for any $y' \in U$ we have $p \in \phi^{-1}(y')$. Thus $\phi^{-1}(U) \subseteq V$. This shows ϕ^{-1} is continuous. $\square$

As there are non-locally-compact complete separable metric spaces and locally compact Hausdorff nonmetrisable spaces, the next theorem is independent of Theorem 2.2. A construction similar to that in the next theorem was used by Lawson [7] to represent complete separable metric spaces in the context of domain theory (see Section 6).

Theorem 2.3. For every complete metric space X there is a poset P such that $X \cong \mathrm{UF}(P)$ and $\mathrm{UF}(P) = \mathrm{MF}(P)$. Moreover, if X is infinite then we may take the cardinality of P to be that of any dense subset of X.

Proof. Let X be a complete metric space; we write $B(x, \epsilon)$ for the open metric ball of radius $\epsilon > 0$ around a point $x \in X$. Let A be a dense subset of X. The poset P is the set of all open balls $B(a, r)$ where r is a positive rational number and $a \in A$. For $p = B(a, r)$ and $p' = B(a', r')$ in P we let $p \prec p'$ if and only if $d(a, a') + r < r'$. An argument similar to the one in the proof of Theorem 2.2 shows that any unbounded filter on P has a unique point in its intersection. The resulting mapping $\phi\colon F \mapsto \bigcap F$ from $\mathrm{UF}(P)$ to X has as its inverse the mapping

$$x \mapsto \{B(a, r) \mid x \in B(a, r), a \in A, r \in \mathbb{Q}^+\}.$$

Each of these mappings can be shown to be continuous using the same method as the proof of Theorem 2.2, using the fact that the open balls included in P form a basis for X. Finally, since X is a complete metric space, every unbounded filter is maximal (see Theorem 3.1 below for details). $\square$

If Theorem 2.3 is applied to the real line using the line itself as the dense subset, the resulting poset P will be uncountable, but $\mathrm{MF}(P) = \mathrm{UF}(P)$ will be homeomorphic to the real line.

There are also second-countable nonmetrisable Hausdorff MF spaces. One example is the Gandy–Harrington space from modern descriptive set theory (see [11]).

3 Separation and countability properties

In this section, we determine the separation properties that a poset space must satisfy. We then show that every second countable poset space is homeomorphic to a poset space obtained from a countable poset. In Section 8, we will show that a countably based poset space is either countable or contains a perfect closed set.

Theorem 3.1. (1) Every UF space is T_0.

(2) Every MF space is T_1.

(3) If $\mathrm{UF}(P)$ is T_1 then every unbounded filter on P is maximal and thus $\mathrm{UF}(P) = \mathrm{MF}(P)$.

Proof. (1) follows from the fact that distinct filters are distinct as subsets of P. (2) follows from the fact that no maximal filter can properly contain another maximal filter. To prove (3), suppose $\mathrm{UF}(P)$ is T_1 and let F be an unbounded filter on P. Let G be a filter on P such that $F \subseteq G$. Clearly G is unbounded. If $F \neq G$ then there must be a $p \in P$ such that $F \in N_p$ and $G \notin N_p$. This means $p \in (F \setminus G)$, which is impossible. Thus $F = G$; this shows that F is maximal. $\square$

Theorem 3.2. Suppose that P is a poset such that $\mathrm{MF}(P)$ is second countable. There is a countable subposet R of P such that the map $F \mapsto R \cap F$ is a homeomorphism from $\mathrm{MF}(P)$ to $\mathrm{MF}(R)$.

Proof. Suppose that $\mathrm{MF}(P)$ is second countable; thus P contains a countable subset Q_0 such that $\{N_q : q \in Q_0\}$ is a basis for the topology, because every basis of a second-countable topology contains a countable subclass which is also a basis. For $n = 0, 1, 2, \ldots$, we construct a set Q_{n+1} inductively to satisfy the following conditions.

- Q_{n+1} is countable.
- $Q_n \subseteq Q_{n+1} \subseteq P$.
- For every $F \in \mathrm{MF}(P)$ and every finite subset $D \subseteq Q_n \cap F$ there is a $q \in Q_{n+1}$ such that $q \preceq d$ for all $d \in D$.

In order to see that Q_{n+1} can be taken to be countable, suppose D is a finite subset of Q_n with nonempty intersection. Let E_D be the set of all $p \in P$ such that $p \preceq d$ for every $d \in D$. For every filter $F \in \mathrm{MF}(P)$ with $D \subseteq F$ there is an element $p \in E_D \cap F$; thus $\{N_e : e \in E_D\}$ is an open cover of the intersection of all open sets N_d with $d \in D$. Since the given space is second countable, there is a countable subset F_D of E_D covering the same set of maximal filters; if some finite subset D of Q_n is not contained in any filter then let F_D be empty. Now take Q_{n+1} to be the union of all F_D where $D \subseteq Q_n$ and D is finite; Q_{n+1} is also at most countable.

Let $R = \bigcup_i Q_i$. Note that R is countable and $\{N_r \mid r \in R\}$ is a basis for $\mathrm{MF}(P)$. For $F \subseteq P$ we write $\phi(F)$ for $F \cap R$. It is straightforward to verify that $\phi(F)$ is a filter for every $F \in \mathrm{MF}(P)$, by the construction of R. Because $R \subseteq P$, every $F \in \mathrm{MF}(R)$ extends to some $F' \in \mathrm{MF}(P)$; then $\phi(F') = F$. This shows that ϕ determines a surjective map Φ from $\mathrm{MF}(P)$ to $\mathrm{MF}(R)$.

In order to prove that Φ is injective, it suffices to prove the following statement. For maximal filters V, W on P we have $V \subseteq W$ if and only if $\phi(V) \subseteq \phi(W)$. Suppose $p \in V \setminus W$. Then $W \notin N_p$ and thus $W \notin N_q$ for all q with $N_q \subseteq N_p$. On the other hand, R is a basis and N_p is the union of basic open sets. Since $V \in N_p$ there is a $r \in R$ with $N_r \subseteq N_p$ and $V \in N_r$. It follows that $r \in \phi(V) \setminus \phi(W)$. The other direction of the implication is trivial.

This shows that ϕ is a bijection from $\mathrm{MF}(P)$ to $\mathrm{MF}(R)$. To see that ϕ is continuous, let $x \in \mathrm{MF}(P)$ be fixed and let U be an open neighborhood of $\phi(x) = x \cap R$ in $\mathrm{MF}(R)$. Without loss of generality we may assume that U is of the form N_r for some $r \in R$. Let $V = \{y \in \mathrm{MF}(P) \mid r \in y\}$ be the basic open set determined by r in $\mathrm{MF}(P)$. Now, because $r \in \phi(x) = x \cap R$, we see that $r \in x$, and thus $x \in V$. Moreover, for any $x' \in V$, we have $r \in x'$, and so $r \in x' \cap R$, which means $\phi(x') \in U$. Thus ϕ is continuous.

To see that ϕ^{-1} is continuous, let V be any open subset of $\mathrm{MF}(P)$, and let $\phi^{-1}(y)$ be in V. Because $\{N_r \subseteq \mathrm{MF}(P) \mid r \in R\}$ is a basis for $\mathrm{MF}(P)$, there is some $r \in R$ with $\phi^{-1}(y) \in N_r \subseteq N_p$. Moreover, any $y' \in \mathrm{MF}(R)$ with $r \in y'$ will satisfy $r \in \phi^{-1}(y')$. Thus, for $U = \{y \in \mathrm{MF}(R) \mid r \in y\}$, we have $y \in U$ and $\phi^{-1}(U) \subseteq V$. This shows that ϕ^{-1} is continuous. □

Corollary 3.3. An MF space is homeomorphic to a countably based MF space if and only if it is second countable.

Corollary 3.4. A UF space is homeomorphic to a countably based UF space if and only if it is second countable.

Proof. Let $X = \mathrm{UF}(P)$ be second countable. Construct a poset R and a map ϕ in a manner analogous to the proof of Theorem 3.2. We show that ϕ is a homeomorphism from $\mathrm{UF}(P)$ to $\mathrm{UF}(R)$. It is clear that if $F \in \mathrm{UF}(P)$ then $\phi(F) \in \mathrm{UF}(R)$. Every $G \in \mathrm{UF}(R)$ extends to some $G' \in \mathrm{UF}(P)$ and then $\phi(G') = G$. Thus ϕ is well defined and surjective as a map from $\mathrm{UF}(P)$ to $\mathrm{UF}(R)$. To see that ϕ is injective, suppose that $F \neq G$ are unbounded filters on P. Without loss of generality we may assume there is some $p \in G \setminus F$. There is thus some r in $R \cap (G \setminus F)$, because R is a basis. But $r \in R \cap (G \setminus F)$ implies $r \in \phi(G) \setminus \phi(F)$, which shows $\phi(G) \neq \phi(F)$. Thus ϕ is a bijection from $\mathrm{UF}(P)$ to $\mathrm{UF}(R)$. The proof that ϕ is a homeomorphism is the same as in the proof of Theorem 3.2. □

4 Product and subspace properties

In this section, we show that the class of MF spaces is closed under taking G_δ subsets and arbitrary topological products. The class of UF spaces is closed under taking G_δ subspaces, but it not closed under even finite products.

Theorem 4.1. The class of MF spaces is closed under arbitrary topological products.

Proof. Suppose that we are given a collection $\langle \langle P_i, \preceq_i \rangle \mid i \in I \rangle$ of posets. We may assume without loss of generality that each poset has a greatest element, which we denote by p_i. We form a poset P consisting of those functions f from I to $\bigcup_{i \in I} P_i$ such that $f(i) \in P_i$ for all i and $f(i) = p_i$ for all but finitely many i. For $f, g \in P$ we put $f \preceq g$ if $f(i) \preceq_i g(i)$ for all i.

We define a map ϕ from $\prod_i \mathrm{MF}(P_i)$ to $\mathrm{MF}(P)$ by sending $\prod_i F_i$ to the set of all functions $f \in P$ such that $f(i) \in F_i$ for all i. The inverse of ϕ takes $x \in \mathrm{MF}(P)$ and returns and returns $\prod_i x_i$, where

$$x_i = \{p \in P_i \mid \text{for some } q \in x, q(i) = p\}.$$

To see that ϕ is continuous, let $x \in \prod_i \mathrm{MF}(P_i)$ be fixed and let U be a basic open neighborhood of $\phi(x)$, so U is of the form N_p for some $p \in P$. Now p is represented by a function $f \colon I \to \bigcup_i P_i$ that returns the maximal element of P_i for all but finitely many $i \in I$. Thus f determines a basic open set V in the product topology $\prod_i \mathrm{MF}(P_i)$ such that V is equal, in each coordinate $i \in I$, to the open set determined by $f(i)$. Then $x \in V$. Suppose

$x' = \prod_i x'_i$ is any point of $\prod_i \mathrm{MF}(P_i)$ that is in V, meaning that $f(i) \in x'_i$ for all $i \in I$. Then $\phi(x')$ will have the property that $p_i \in x'_i$ for each $i \in I$, which means $\phi(x') \in N_p$. Thus ϕ is continuous.

To see that ϕ^{-1} is continuous, let $y \in \mathrm{MF}(P)$ be fixed, and let V be any neighborhood of $\phi^{-1}(y)$ in $\prod_i \mathrm{MF}(P_i)$. By the definition of the product topology, there is a basic open neighborhood of $\phi^{-1}(y)$ which is obtained as a product $\prod_i V_i$ of open sets $V_i \subseteq \mathrm{MF}(P_i)$ such that $V_i = \mathrm{MF}(P_i)$ for all but finitely many $i \in I$. Moreover, in the finitely many coordinates where V_i is a proper subset of $\mathrm{MF}(P_i)$, we can find a basic open subset $N_{r(i)} \subseteq V_i$ such that the projection of $\phi^{-1}(y)$ to coordinate i is in $N_{r(i)}$. For all i where $V_i = \mathrm{MF}(P_i)$ we let $r(i)$ be the greatest element of P_i. Now let f be the element of P such that $f(i) = r(i)$ for all $i \in I$. Then $y \in N_f$ (in $\mathrm{MF}(P)$), and any $y' \in N_f$ will satisfy $\phi^{-1}(y) \in V$. Thus ϕ^{-1} is continuous. □

Corollary 4.2. Every topological product of countably many countably based MF spaces is homeomorphic to a countably based MF space.

Proof. Under these hypotheses, the poset constructed in Theorem 4.1 is countable. □

Theorem 4.3. The class of MF spaces is closed under taking G_δ subspaces.

Proof. Suppose that $\langle U_i \mid i \in \mathbb{N}\rangle$ is a sequence of open subsets of $\mathrm{MF}(P)$ and $U = \bigcap_i U_i$ is nonempty. We form a poset Q of pairs $\langle n, p\rangle$ such that $n \in \mathbb{N}$ and $N_p \subseteq \bigcap_{i<n} U_i$, declaring $\langle n, p\rangle \prec \langle n', p'\rangle$ if $n > n'$ and $p \preceq p'$. We define a map ϕ from $\bigcap_i U_i$ to $\mathrm{MF}(Q)$ by sending a maximal filter F to the set of all $\langle n, p\rangle$ in Q such that $F \in N_p$. The inverse ψ of ϕ takes a maximal filter $G \in \mathrm{MF}(Q)$ and returns the set

$$\psi(G) = \{p \in P \mid \text{for some } n \in \mathbb{N}, \langle n, p\rangle \in x\}$$

To see that $\psi(G)$ is a filter, note that if $\langle n, p\rangle \in G$ and $\langle m, q\rangle \in G$ then there is some common extension $\langle o, r\rangle \in G$, and thus r is a common extension of p and q in $\psi(G)$.

To see that $\psi(G)$ is maximal, note that if $\bigcap\{p \mid p \in \psi(G)\}$ contained more than one point of U, then at least one of the points of the intersection has a basic open neighborhood N_q that does not contain one other point of the intersection. It is then possible to adjoin N_q to G and extend this to a filter, contradicting the maximality of G.

To see that ϕ is continuous, note that if $\phi(F) \in \langle n, p\rangle$ then for every $F' \in \mathrm{MF}(P) \cap N_p$, we have $\phi(F) \in \langle n, p\rangle$. Conversely, if $\psi(G) \in N_p \cap U$ then $\psi(G) \in U_1$ and thus every $G' \in N_{\langle 1,p\rangle} \subseteq \mathrm{MF}(Q)$ will have $\psi(G') \in N_p$. $\square$

Theorem 4.3 gives an optimal result. We will show below that all poset spaces have the property of Baire. The real line is homeomorphic to a UF space, but the F_σ subset of rational numbers does not have the property of Baire and therefore is not homeomorphic to a poset space.

The class of UF spaces does not enjoy the closure properties that the class of MF spaces does. We now give an example showing that the class of UF spaces is not closed under finite products.

Example 4.4. There are two posets P, Q such that $\mathrm{MF}(P) = \mathrm{UF}(P)$, $\mathrm{MF}(Q) = \mathrm{UF}(Q)$, but the topological product $\mathrm{MF}(P) \times \mathrm{MF}(Q)$ is not homeomorphic to any UF space.

Proof. Let ω denote the least infinite countable ordinal and let ω_1 denote the least uncountable ordinal. We define P to be the set of functions from finite initial segments of ω to $\{0, 1\}$ and define Q to be the set of functions from countable initial segments of ω_1 to $\{0, 1\}$. For both posets the relation $\preceq$ is given by extension: $p \preceq q$ if, for all α in the domain of q, $p(\alpha)$ is defined and takes the value $q(\alpha)$.

We first show that $\mathrm{MF}(P) = \mathrm{UF}(P)$ and $\mathrm{MF}(Q) = \mathrm{UF}(Q)$. Assume that F is an unbounded filter on P (the argument for Q is parallel). Then all functions in F are compatible, that is, they do not contradict each other on any value in the intersection of their domains. There is thus a total limit function f, because otherwise there would be a first ordinal α where f is undefined and the function extending f which maps α to 0 would define an element of P which would be a lower bound for the filter F. Since f is total, all functions mapping the ordinals up to some α in the domain of f to the corresponding value of f are in the filter. One can see that this filter is already maximal, because any element outside it but still in P is incompatible with this function and adding it would destroy the filter property.

Assume now, by way of contradiction, that $\mathrm{UF}(P) \times \mathrm{UF}(Q)$ is homeomorphic to a space $\mathrm{UF}(R)$. We denote by π_P, π_Q the continuous, open projection maps from $\mathrm{UF}(R)$ to its factor spaces. There is a filter F in $\mathrm{UF}(R)$ such that $\pi_P(F)$ and $\pi_Q(F)$ are the filters generated by the set of all functions in P and Q, respectively, which map all inputs to 0. Now one can select an infinite sequence $r_0, r_1, \ldots$ in F such that for each n the projection $\pi_P(N_{r_n})$

consists only of functions which map the first n numbers to 0 and $r_{n+1} \preceq r_n$ for all n. The sequence $\langle r_i \rangle$ generates a subfilter $G \subseteq F$. There is no lower bound r for G, because otherwise $\pi_P(N_r)$ would be an open set containing some basic open set N_p such that $N_p \subseteq \pi_P(N_{r_n})$ for all n; such a p cannot exist by construction.

On the other hand, there is a function f contained in all the open sets $\pi_Q(N_{r_n})$ and there are basic open neighbourhoods of f generated by $q_0, q_1, \ldots$ such that $N_{q_n} \subseteq \pi_Q(N_{r_n})$ for each n. The basic open sets $N_{q_0}, N_{q_1}, \ldots$ fix f only on countably many ordinals and thus their intersection is also a basic open set. So $\pi_Q(G)$ is bounded while $\pi_Q(F)$ is not and thus $G \subset F$. It follows that $\mathrm{UF}(R)$ is not a T_1 space. This contradicts the assumption that $\mathrm{UF}(R)$ is homeomorphic to $\mathrm{MF}(P) \times \mathrm{MF}(Q)$. □

We note that the previous example is not second countable and that the failure of second countability was important to the proof.

Question 4.5. Is the class of countably based UF spaces closed under taking finite (or arbitrary) topological products?

We end the section by showing that the class of UF spaces is closed under taking G_δ subspaces. As with the class of MF spaces, this result cannot be extended to include F_σ subspaces. We first prove the result for open subspaces, which has a much simpler proof.

Theorem 4.6. The class of UF spaces is closed under taking open subspaces.

Proof. Let P be a poset and let U be an open subset of $\mathrm{UF}(P)$. Let R be the set of all $r \in P$ such that $N_r \subseteq U$; we regard R as a subposet of P. Then any $x \in U$ has a neighborhood $N_r \subseteq U$, where $r \in R$. Thus the restriction map $\phi\colon x \mapsto x \cap R$ sends each element of U to a filter on R. Note that if this filter were not unbounded as a subset of R then it has a lower bound in R and consequently would not be unbounded in P.

The inverse map of ϕ sends each maximal filter on R to it upward closure in P. If $\phi(G)$ were bounded below by $p \in P$, then in particular $p \preceq r$ for some $r \in R$. Thus $N_p \subseteq N_r \subseteq R$, which means $p \in R$ and G is not in $\mathrm{UF}(R)$.

To set that ϕ and its inverse are continuous, note that

$$\left\{ N_r^P = \{F \in \mathrm{MF}(P) \mid r \in F\} \;\mid r \in R \right\}$$

is a basis for the restriction of $\mathrm{MF}(P)$ to the subspace U, that

$$\left\{ N_r^R = \{F \in \mathrm{MF}(R) \mid r \in F\} \;\mid r \in R \right\}$$

is a basis for $\mathrm{MF}(R)$, and that a point $x \in U \subseteq \mathrm{MF}(P)$ is in N_r^P if and only if $\phi(x)$ is in N_r^R. $\square$

Theorem 4.7. The class of UF spaces is closed under taking G_δ subspaces.

Proof. Let G_0 be the space $\mathrm{UF}(P)$ for some poset P with order $\prec_P$ and let G be a G_δ subset of G_0. Thus there is a descending sequence $G_1, G_2, \ldots$ of open subsets of G_0 such that $G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ and $G = \bigcap_n G_n$. Define

$$R = \{p \in P \mid p \in F \text{ for some } F \in \mathrm{UF}(P) \cap G\}.$$

For each $p \in R$, let $g(p) = \sup\{n \in \mathbb{N} \mid N_p \subseteq G_n\}$, where $g(p) = \infty$ if $N_p \subseteq G$. Define an order relation $\prec_R$ on R by putting $p \prec_R q$ if $p \prec_P q$ and either $g(q) < g(p) \leq \infty$ or $g(q) = g(p) = \infty$. We will show that the unbounded filters on $(R, \preceq_R)$ are precisely the unbounded filters on P that are in G and do this by showing the following four claims.

Claim 1: Let $F \in G \subseteq \mathrm{UF}(P)$; then F is an unbounded filter in R under $\preceq_R$. By definition of R, $F \subseteq R$. To show that F is a filter on R, fix $p, q \in F$. If $g(p)$ or $g(q)$ is infinite then p and q have a common extension r under $\preceq_P$ with $g(r) = \infty$. Thus r is a common extension of p and q under $\preceq_R$. Otherwise, because $F \in G$, there is an $r \in F$ with $r \preceq_P p$, $r \preceq_P q$ and $N_r \subseteq G_{g(p)+g(q)+1}$. Then $g(r) > g(p) + g(q)$, $r \prec_R p$ and $r \prec_R q$. As $\preceq_R$ is a restriction of $\preceq_P$, F is upward closed under $\preceq_R$ and F is a filter in R. Furthermore, F must be unbounded in R, because a bound in R would also be in a bound P.

Claim 2: Let $F \subseteq R$ be a filter in R; then either $\sup\{g(p) \mid p \in F\} = \infty$ or F is bounded. Suppose the supremum is $n < \infty$ instead. There can only be one $r \in F$ with $g(r) = n$, because F is a filter on R. Because $r \in R$, there is some $F' \in \mathrm{UF}(P)$ with $r \in F'$ and $F' \in G$. Thus there is an $r' \in F'$ with $g(r') > g(r)$ and $r' \prec_P r$; this means $r' \prec_R r$, which shows that F is bounded in R.

Claim 3: Let F be a bounded filter of P which is also a filter in R; then F is bounded in R. Let $r \in P$ be a lower bound for F. If $N_r \not\subseteq G_n$ for some n then $\sup\{g(p) \mid p \in F\} < n$ and F is bounded in R by Claim 2. Otherwise $N_r \subseteq G$, in which case $r \in R$ and F is again bounded as a subset of R.

Claim 4: Let F be an unbounded filter in R; then F is also an unbounded filter in P. To see this, consider the upward closure F' of F in P. F' is unbounded in P, by Claim 3. Claim 2 shows that $F' \in G$; thus $F' \subseteq R$. The definition of F' shows that $F \subseteq F'$. Fix $r \in F'$; then there must be a $p \in F$

with $p \preceq_P r$. If $g(p) = \infty$ then $p \preceq_R r$ and so $r \in F$. Otherwise there must be a $q \in F$ with $q \preceq_P p$ and $g(q) > g(r)$. Then it follows from transitivity of $\preceq_P$ and the definition of $\prec_R$ that $q \prec_P r$, $q \prec_R r$ and $r \in F$. This shows $F' = F$.

Claims 1 and 4 show that the unbounded filters on R are exactly those unbounded filters on P which are in G. So the identity map $\phi\colon \mathrm{UF}(P) \cap G \to \mathrm{UF}(R)$ is surjective by Claim 4. As this map is trivially injective, it is thus invertible. To see that ϕ and ϕ^{-1} are continuous, let $x \in \mathrm{UF}(P) \cap G$ be fixed. Note that for any $r \in R$, we have $r \in x$ if and only if $r \in \phi(x)$, because ϕ is the identity map on filters. Thus $\phi(x)$ is in the basic open neighorhood of $\mathrm{UF}(R)$ determined by r if and only if x is in the basic open neighborhood of $\mathrm{UF}(P) \cap G$ determined by r. □

5 Completeness properties

In this section, we establish that every poset space has the a completeness property known as the strong Choquet property. We then characterize the class of countably based MF spaces as precisely the class of second-countable T_1 spaces with the strong Choquet property. We first establish a weaker property.

Theorem 5.1. Every poset space has the property of Baire.

Proof. Let X be $\mathrm{MF}(P)$ or $\mathrm{UF}(P)$. Suppose that $\langle U_i \mid i \in \mathbb{N}\rangle$ is a sequence of dense open sets in X and V is a fixed open set. We construct a sequence $\langle p_i \mid i \in \mathbb{N}\rangle$ of elements of P. Let p_0 be such that $N_{p_0} \subseteq V \cap U_0$. Given p_i, there is an unbounded or maximal filter in $N_{p_i} \cap U_{i+1}$. Choose p_{i+1} such that $N_{p_{i+1}} \subseteq U_{i+1} \cap N_{p_i}$ and $p_{i+1} \preceq p_i$. In the end, $F = \langle p_i \rangle$ is a linearly ordered subset of P. Thus F extends to an element of X. Clearly this element is in $V \cap \bigcap_i U_i$. □

We will now show that every poset space has the strong Choquet property, which is defined using a certain game first introduced by Choquet [1]. Let X be an arbitrary topological space. The strong Choquet game is the Gale–Stewart game (see [4] and [6]) defined as follows. The stages of play are numbered $0, 1, 2, \ldots$ and both players make a move in each stage. In stage i, player I plays an open set U_i and a point x_i such that $x_i \in U_i$ and if $i > 0$ then $U_i \subseteq V_{i-1}$. Then player II plays an open set V_i such that $x_i \in V_i$ and $V_i \subseteq U_i$.

At the end of the game, player I wins if $\bigcap_i U_i$ is empty (or, equivalently, if $\bigcap_i V_i$ is empty). Player II wins if $\bigcap_i U_i$ is nonempty. A *position* in the game is a finite (possibly empty) sequence

$$\langle \langle U_0, x_0 \rangle, V_0, \langle U_1, x_1 \rangle, \ldots \rangle$$

which is an initial segment of an infinite play of the game following the rules just described.

A space X has the *strong Choquet property* if player II has a winning strategy for the strong Choquet game on X. A winning strategy is a function that takes a position after player I has played and tells player II which open set to play, such that if player II follows the winning strategy then player II will always win the game regardless of what moves player I makes.

The strong Choquet property is strictly stronger than the property of Baire. Moreover, the class of topological spaces with the strong Choquet property is closed under G_δ subspaces and arbitrary topological products. It is known that the class of topological spaces with the property of Baire is not closed under binary products (an example is provided in [3]).

Theorem 5.2. Every poset space has the strong Choquet property.

Proof. We describe the strategy for player II informally. At the start of the game, player I plays an open set U_0 and a point x_0. Player II translates the point x_0 into a filter on P, then finds a basic neighbourhood q_0 of x such that $N_{q_0} \subseteq U_0$. Player II then plays N_{q_0}. Now given $\langle x_1, U_1 \rangle$ with $x_1 \in N_{q_0}$, Player II translates x_1 to a filter on P and then finds a neighbourhood q_1 of x_1 such that $q_1 \preceq_P q_0$ and $N_{q_1} \subseteq U_1$. Player II plays N_{q_1}. Player II continues this strategy, always choosing $q_{i+1} \preceq_P q_i$. At the end of the game, player II has determined $\{q_i \mid i \in \mathbb{N}\}$, a descending sequence of elements of P. This sequence extends to an element of X which is in $\bigcap N_{q_i}$. Player II has thus won the game. $\square$

We use the strong Choquet property to obtain the following characterization of countably based MF spaces.

Theorem 5.3. A topological space is homeomorphic to a countably based MF space if and only if it is second countable, T_1 and has the strong Choquet property.

We postpone the proof of this theorem temporarily to comment on the hypotheses involved in the characterization. Clearly, any space X homeomorphic to a countably based MF space must be T_1 and second countable. We have already shown X must also have the strong Choquet property. Thus the new content of Theorem 5.3 is that the strong Choquet property is sufficient for a T_1 second-countable space to be homeomorphic to a countably based MF space. In the non-second-countable setting, the strong Choquet property is not sufficient for a T_1 space to be homeomorphic to an MF space.

Example 5.4. There is a Hausdorff strong Choquet space which is not homeomorphic to any MF space.

Proof. The space X consists of certain functions from ω_1 to $\{0, 1\}$. We put a function f in X if and only if there is an ordinal $\alpha < \omega_1$ such that $f(\beta) = 0$ for all $\beta > \alpha$. For each $f \in X$ and each $\alpha < \omega_1$, the set

$$\{g \in X \mid f(\beta) = g(\beta) \text{ for all } \beta < \alpha\}$$

is declared to be an open set. The topology on X is the one generated by these open sets. It is clear that X is a Hausdorff space.

It is easy to show that X has the strong Choquet property, as follows. All that player II has to do is to play any basic open subset of the open set played by player I which also contains the point given by player I. In the end, the open sets played by player I in the countable number of rounds of the game and each round will fix countably many coordinates of a function in X. In the limit, countably many coordinates are fixed and we can find a point in the intersection of the sets played by I by forcing the remaining coordinates to map to 0.

We now show that X is not homeomorphic to any MF space. Suppose, by way of contradiction, that $X \cong \mathrm{MF}(P)$. We construct a transfinite sequence $\langle p_\alpha \mid \alpha < \omega_1\rangle$ inductively. Let p_0 be any basic open neighbourhood of the constant 0 function. Given $\langle p_\alpha \mid \alpha < \beta\rangle$, there is a first coordinate $\gamma < \omega_1$ which is not fixed by any p_α; let f be the function which is 0 except at γ, and $f(\gamma) = 1$. Note that any intersection of countably many open sets in X is open. Thus we may choose $p_\beta \in P$ such that $p_\beta \preceq p_\alpha$ for all $\alpha < \beta$ and $f \in N_{p_\beta}$. Choose any such p_β. At the end of this construction, $\langle p_\alpha \mid \alpha < \omega_1\rangle$ is linearly ordered and thus extends to a maximal filter F. Now the element of X corresponding to F sends uncountably many ordinals to 1, which is impossible. $\square$

We now return to the proof of Theorem 5.3, which will occupy the remainder of this section. Let X be a fixed T_1 space with a fixed countable basis and a fixed winning strategy for player II in the strong Choquet game. Our first step is to define a poset P. The elements of P are called *conditions*. A condition is a finite list of the form

$$\langle A, \pi_1, \pi_2, \ldots, \pi_k \rangle$$

satisfying the following requirements.

(1) The set A is a nonempty basic open set from the fixed countable basis. For each condition c we let $S(c)$ denote the basic open set A appearing in c.

(2) Each π_i is a finite (that is, partial) play of the strong Choquet game on X following the fixed winning strategy s_{II} for player II. We require each π_i to be of the form

$$\langle V_1, x_1, s_{\mathrm{II}}(V_1, x_1), V_2, x_2, s_{\mathrm{II}}(V_1, x_1, V_2, x_2), \ldots,$$
$$V_r, x_r, s_{\mathrm{II}}(V_1, x_1, V_2, x_2, \ldots, V_r, x_r) \rangle.$$

Thus each π_i ends with an open set, which we will denote by $U(\pi_i)$. It is allowable that π is the empty sequence $\langle\rangle$, in which case $U(\pi) = X$.

(3) If a play π is in a condition then so is every initial segment of π that ends with a move by player II.

(4) $A \subseteq U(\pi_i)$ for each $i \leq k$.

We define the order $\prec$ on P as follows. Let $c = \langle A, \pi_1, \pi_2, \ldots, \pi_k \rangle$ and $c' = \langle A', \pi'_1, \pi'_2, \ldots, \pi'_l \rangle$ be any two conditions. We let $c' \prec c$ if and only if

(5) For each finite play π_i in c there is a point $x_n \in S(c)$ such that the longer play

$$\pi_i \frown \langle A, x_n, s_{\mathrm{II}}(\pi_i \frown \langle A, x_n \rangle) \rangle$$

is in c', that is, equals π'_j for some $j \leq l$.

(6) $A' \subseteq A$ (this is actually a consequence of requirement (5)).

Requirement (3) in the definition of a condition allows us to prove that the order on P is transitive. Because each condition is finite, requirement (5) in the definition of the order relation ensures $c \not\prec c$ for all $c \in P$. Thus $\prec$ is a partial order on P.

Lemma 5.5. For any filter F on P the intersection $\bigcap_{c \in F} S(c)$ is nonempty.

Proof. Let $\langle A_i \mid i \in \mathbb{N} \rangle$ be an enumeration of all of the basic open sets which appear as $S(c)$ for some $c \in F$; here we are using the fact that X is second countable and that each $S(c)$ is drawn from a fixed countable basis of X. It is immediate that $\bigcap_{c \in F} S(c)$ equals $\bigcap_{i \in \mathbb{N}} A_i$. We will show the latter intersection is nonempty.

We inductively construct a descending sequence of conditions $\langle c_i \mid i \in \mathbb{N} \rangle$ and a sequence of finite plays $\langle \pi_i \mid i \in \mathbb{N} \rangle$ so that π_{i+1} is an immediate extension of π_i for each $i \in \mathbb{N}$. At stage 0 let c_0 be any condition in F such that $S(c_0) = A_0$ and let π_0 be any finite play in c_0.

At stage $i + 1$ let c be any condition in F such that $S(c) = A_i$. Let c_{i+1} be a common extension of c and c_i in F. It is clear that $S(c_{i+1}) \subseteq S(c) = A_i$. Choose π_{i+1} to be any play in c_{i+1} which is an immediate extension of π_i.

Now assume the entire sequence $\langle \pi_i \rangle$ has been constructed. These partial plays determine an infinite play γ of the strong Choquet game following the strategy for player II. Thus the intersection of the open sets played by player I in γ is nonempty. By construction, each set A_i has a subset played by player I at some stage of γ. Thus $\bigcap_i A_i$ is nonempty. $\square$

Lemma 5.6. Let c_1 and c_2 be two conditions and let $x \in S(c_1) \cap S(c_2)$. There is a condition c such that $c \prec c_1$, $c \prec c_2$ and $x \in S(c)$.

Proof. Begin by letting c be empty. For each π in c_1 we put the longer play

$$\pi \frown \langle S(c_1), x, s_{\mathrm{II}}(\pi \frown \langle S(c_1), x \rangle) \rangle$$

into c. For each π in c_2 we put

$$\pi \frown \langle S(c_2), x, s_{\mathrm{II}}(\pi \frown \langle S(c_2), x \rangle) \rangle$$

into c. For each π that has been added to c we add all initial segments of π ending with a move by player II. We then let $S(c)$ be a basic open neighbourhood of x which is a subset of the open set $\bigcap_{\pi \in c} U(\pi)$. This construction ensures that c is a condition satisfying the conclusions of the lemma. $\square$

Lemma 5.7. Let F be a maximal filter on P. The intersection $\bigcap_{c \in F} S(c)$ contains a single point.

Proof. By Lemma 5.5 we know that $\bigcap_{c\in F} S(c)$ is nonempty. Suppose that x, y are distinct points in $\bigcap_{c\in F} S(c)$. Let A be a basic open neighbourhood of x such that $y \notin A$. We construct a filter G inductively. At stage n we construct $G_n \subseteq P$ and in the end we let G be the upward closure of $\bigcup_n G_n$. To begin, let $G_0 = F \cup \{\langle A, \langle\rangle\rangle\}$. At stage $i+1$, we know by induction that $x \in S(c)$ for every $c \in G_i$. Thus we can apply Lemma 5.6 repeatedly so that $G_i \subseteq G_{i+1}$, every pair of conditions in G_i has a common extension in G_{i+1} and $x \in S(c)$ for every $c \in G_{i+1}$.

It is immediate from the construction that $G = \bigcup_i G_i$ is a filter which properly extends F. This shows that F was not maximal. $\square$

Proof of Theorem 5.3. For each $F \in \mathrm{MF}(P)$ we denote the single point in $\bigcap_{c\in F} S(c)$ by $\phi(F)$. We show that ϕ is a homeomorphism from $\mathrm{MF}(P)$ to X.

We first show that ϕ is an injective map. Suppose that F and F' are maximal filters on P such that $x \in \bigcap_{c\in F} S(c)$ and $x \in \bigcap_{c\in F'} S(c)$. By following a procedure similar to the proof of Lemma 5.7 we may find a filter G such that $F \subseteq G$ and $F' \subseteq G$. Thus, by maximality, we have $F = F' = G$.

We next show that ϕ is a surjective map. Let $x \in X$ be fixed. Let $\langle A_i \mid i \in \mathbb{N}\rangle$ be a sequence of basic open sets such that $\bigcap_i A_i = \{x\}$. The existence of this sequence requires that X be T_1 and first countable. For each $i \in \mathbb{N}$ let $c_i = \langle A_i, \langle\rangle\rangle$. Following a method similar to the proof of Lemma 5.7, we can construct a filter F such that $c_i \in F$ for each $i \in \mathbb{N}$. Let G be an extension of F to a maximal filter. Now $S = \bigcap_{c\in G} S(c)$ is nonempty by Lemma 5.5 and $S \subseteq \bigcap_i A_i = \{x\}$ by construction, so $\phi(G) = x$.

It remains to show that ϕ is open and continuous. This follows from Lemma 5.6; for each $x \in X$ and each condition c, we have $c \in \phi^{-1}(x)$ if and only if $x \in S(c)$. This shows that X is homeomorphic to $\mathrm{MF}(P)$. By Theorem 3.2, we may find a countable subposet R of P such that X is homeomorphic to $\mathrm{MF}(R)$. This completes the proof. $\square$

6 An application to domain theory

In this section, we apply the characterization of countably based MF spaces to characterize those second-countable spaces with a domain representation. Our result gives a complete solution to the so-called model problem for second-countable spaces in domain theory.

A domain is a certain type of poset (defined below) and every domain is a topological space with a topology known as the Scott topology. A domain representation of a topological space X is a domain D such that X is homeomorphic to the topological space consisting of the maximal elements of D with the relative Scott topology. The history of such representations is thoroughly described by Martin [8]. It is known that every complete separable metric space has a domain representation (see Lawson [7]) and that every space with a domain representation is T_1 and has the strong Choquet property (Martin [8]). We now show that the strong Choquet property is sufficient for a T_1 second-countable space to have a domain representation.

We summarize the definitions from domain theory that we require; these definitions are explored fully by Gierz *et al.* [5]. A nonempty subset I of a poset $\langle P, \preceq \rangle$ is *directed* if every pair of elements in I has an upper bound in I. A poset P is said to be a *dcpo* (for "directed-complete partial ordering") if every directed subset of P has a least upper bound. Any dcpo D has a second order relation $\ll$, known as the *way below* relation, under which $q \ll p$ if and only if whenever $I \subseteq D$ is a directed set with $p \ll \sup I$ there is some $r \in I$ with $q \preceq r$. For each $p \in D$ we put $\Downarrow p = \{q \in D \mid q \ll p\}$ and $\Uparrow q = \{p \in D \mid q \ll p\}$. A dcpo D is *continuous* if $\Downarrow p$ is directed and the equality $p = \sup \Downarrow p$ holds for every $p \in D$. A *domain* is a continuous dcpo. A subset B of a domain D is a *basis* if $B \cap \Downarrow p$ is directed and $p = \sup(B \cap \Downarrow p)$ for every $p \in D$. A domain is *ω-continuous* if it has a countable basis. An element p of a dcpo is *compact* if $p \ll p$. A dcpo D is *ω-algebraic* if there is a countable basis for D consisting of compact elements. The *Scott topology* on a dcpo D is generated by the basis $\{\Uparrow p \mid p \in D\}$. A *domain representation* of a space X is a homeomorphism between X and the maximal elements of a domain with the Scott topology.

Theorem 6.1. A topological space has a domain representation via an ω-algebraic dcpo if and only if the space is second-countable, T_1 and has the strong Choquet property.

Proof. It can be seen that any space with a domain representation satisfies the T_1 separation property and a result of Martin [8] shows that any space with a domain representation has the strong Choquet property. Therefore, we only need to prove that a second-countable T_1 strong Choquet space has a domain representation via an ω-algebraic dcpo We use the following lemma, which follows easily from the definitions.

Lemma 6.2. Suppose that P is a countable poset. The set of all filters on P, ordered by inclusion, is an ω-algebraic dcpo D. The maximal filters on P are precisely the maximal elements of D and the compact elements of D are precisely the principal filters on P. Moreover, the poset topology on $\mathrm{MF}(P)$ corresponds exactly to the Scott topology on the maximal elements of D.

We showed in Theorem 5.3 that any second-countable T_1 strong Choquet space is homeomorphic to $\mathrm{MF}(P)$ for a countable poset P. It follows immediately from the lemma above that such a space also has a domain representation via an ω-algebraic dcpo. $\square$

The next corollary follows from the fact that any space with a domain representation is T_1 and has the strong Choquet property. Although this corollary is already known, the proof here is new.

Corollary 6.3. If a second-countable space has a domain representation then it has a representation via an ω-algebraic dcpo.

We end this section with several remarks on the relationship between domain representable spaces and MF spaces.

A proof of Lemma 6.2 can be modified to show that the collection of all ideals on a poset (sometimes called the *ideal completion* of the poset) forms a domain whose maximal elements in the Scott topology correspond to the maximal ideals of the poset under the Stone topology. All results we have proved for MF spaces also hold for these spaces of maximal ideals, by duality. The relationship between ideal completions and domain representations has been investigated by Martin [9].

A *Scott domain* is a domain in which every pair of elements with an upper bound has a least upper bound. Lawson [7] has shown that any space with a domain representation via a countably based Scott domain is a complete separable metric space. It can be seen that posets constructed in Theorem 5.3 do not, in general, give Scott domains, even when the posets are constructed from formal balls in complete separable metric spaces.

The proof of Example 5.4 can be modifed to obtain the following.

Example 6.4. There is a Hausdorff strong Choquet space that does not have a domain representation.

7 Semi-Topogenous Orders

In this section, we prove results which give a partial solution to the question of which arbitrary (not necessarily second countable) topological spaces are homeomorphic to MF spaces.

Suppose that a topological space X is homeomorphic to $\mathrm{MF}(P)$, for some poset P, via a fixed homeomorphism ϕ. If each element of $p \in P$ is replaced by the corresponding open subset $\phi(N_p) \subseteq X$, the poset order on P will determine a corresponding order relation on these subsets of X. Moreover, the collection of all these open subsets forms a basis for the topology on X. It is thus natural to ask whether the existence of a basis with a suitable order relation is sufficient for a topological space to be homeomorphic to an MF space.

Császár [2] considered many different types of orders and their connections to topology. The basic concept is that of a semi-topogenous order.

Definition 7.1. A *semi-topogenous order* is a binary relation $\sqsubset$ on the powerset of a topological space X satisfying the following axioms for all $u, v, w \subseteq X$ [2, Chapter 2]:

- $\varnothing \sqsubset \varnothing$ and $X \sqsubset X$;
- $v \sqsubset w \Rightarrow v \subseteq w$;
- $u \subseteq v \sqsubset w \Rightarrow u \sqsubset w$;
- $u \sqsubset v \subseteq w \Rightarrow u \sqsubset w$.

Császár considered orders which are only linked to topology, such as the order which says that w is a neighbourhood of v. It might happen that some but not all open supersets w of a given set v satisfy $v \sqsubset w$. Nevertheless, although this is not made explicit by Császár, it is quite convenient to postulate also a connection between the topology and the open spaces.

Recall that the *open kernel* of a set is the union of all its open subsets. We say that the topological space X is *generated* by the order $\sqsubset$ if for each $u \subseteq X$ the set $\bigcup\{o \subseteq X \mid o \sqsubset u\}$ is the open kernel of u. In this case a set w is open if and only if it is the union of all v such that $v \sqsubset w$. It follows that if $v \sqsubset w$ then there is an open o with $v \subseteq o \subseteq w$; the converse of this last implication does not always hold. Every topological space is generated by some semi-topogenous order, for one can define $v \sqsubset w$ to hold if and only if there is an open set o with $v \subseteq o \subseteq w$.

Remark 7.2. There is a close relationship between semi-topogenous orders and the way below relation $\ll$ on a continuous dcpo, which was discussed in Section 6. The following true properties of the way below relation are obtained by dualizing the second, third and fourth properties in the definition of a semi-topogenous order:

$$v \ll w \Rightarrow v \leq w$$
$$u \leq v \ll w \Rightarrow u \ll w$$
$$u \ll v \leq w \Rightarrow u \ll w$$

The fact that these are dual forms follows from the fact that points in a topological space are minimal as nonempty subsets under $\subseteq$ but are maximal elements of a domain representing the topological space; for this reason, we write $\leq$ for $\supseteq$ and $\ll$ for $\sqsupset$. The requirement that $\bigcup\{o \mid o \sqsubset u\}$ is the open kernel of u corresponds exactly to the fact that $\{x \mid y \ll x\}$ is the open kernel of an element y of a continuous dcpo with the Scott topology.

Thus if a space X has a representation via a continuous dcpo D then the dual of the way below relation on D is a semi-topogenous order (except that it is defined only on a subset of the powerset of X) which generates the topology on X. Semi-topogenous orders can be viewed as a generalization of the way below relation which is applicable to the case when the dcpo is the full powerset of a topological space. It appears that semi-topogenous orders are related to auxilliary relations as defined by Gierz *et al.* [5], although a formal relationship seems difficult to state.

A *filter* in a topological space X is a collection of nonempty subsets that is closed under finite intersection and under superset. A filter *has an open basis* if for every w there is an open v in the filter with $v \subseteq w$. As in general there need not be a point contained in the intersections of the sets in a filter, we are interested in a condition on filters that requires their sets to contain a common point. Our condition that a filter meets a semi-topogenous order will imply that this filter has also an open basis, while a compelteness condition will ensure that each filter that meets the order has a nonempty intersection.

Definition 7.3. Let X a space with a semi-topogenous order $\sqsubset$ generating its topology. A filter U on X *meets* $\sqsubset$ if for every $w \in U$ there is a $v \in U$ with $v \sqsubset w$. X is *complete for* $\sqsubset$ if for every filter U in the space X which meets $\sqsubset$ there is a point x with $x \in u$ for all $u \in U$.

Theorem 7.4. Let X be a T_1 space with a semi-topogenous order $\sqsubset$ generating its topology such that X is complete for $\sqsubset$. Then X is homeomorphic to an MF space.

Proof. Let P consist of the nonempty open subsets of X and let $p \prec q$ if and only if $p \neq q$ and $p \sqsubset q$. The relation $\prec$ is obviously transitive and antireflexive, and so it makes P into a poset.

For each $x \in X$ let U_x be the set of all $p \in P$ with $\{x\} \sqsubset p$. If $p, q \in U_x$ then the open kernel u of $p \cap q$ contains x and thus there is an open $r \sqsubset u$ with $x \in r$. As the open kernel of r again contains x, $\{x\} \sqsubset r$. So $r \in U_x$, $r \sqsubset p$ and $r \sqsubset q$. Thus U_x is a filter on P.

If V is a maximal filter on $\mathrm{MF}(P)$ then V also meets $\sqsubset$. If v generates V then v is open (by definition) and not empty. For every $x \in v$ there is a $w \sqsubset v$ with $x \in w$; by maximality $w = v$. Thus $v \sqsubset v$ and every $w \subseteq X$ with $v \subseteq w$ satisfies $v \sqsubset w$ and $w \in V$. If there is no single element generating V then there is, for every $v, w \in V$, some $u \in V$ with $u \prec v$ and $u \prec w$. Then it follows that $u \sqsubset v$ and $u \sqsubset w$. Furthermore, there is an $t \prec u$ with $t \in V$; then it follows that $t \sqsubset v \cap w$. So V contains all supersets of $v \cap w$ and thus V is a filter. Furthermore, V meets $\sqsubset$.

This means, by assumption, that there is a point x contained in all sets of V. Thus $V \subseteq U_x$ and by the maximality of V, $V = U_x$. Therefore, every filter U_y is contained in a filter U_x which is maximal. Due to the T_1 property, $y = x$; otherwise U_y would contain a p with $x \notin p$ in contradiction to the fact that $U_y \subseteq U_x$.

This shows that the mapping $\phi\colon x \mapsto U_x$ is a bijection from X to the maximal filters on P. To see the ϕ is open and continuous, first note that if $y \in X$ and U is an open set, then $y \in U$ if and only if $\{y\} \sqsubset U$. To see this, fix $y \in X$ and any open U such that $y \in U$, which means then $\{y\} \subseteq U$. Then, because $\sqsubset$ generates the topology and y is trivially in the open kernel of U, there is some $W \sqsubseteq U$ with $y \in W$. This means $\{y\} \subseteq W \sqsubset U$, which means $\{y\} \sqsubset U$ by the definition of semi-topogenous orders. The converse direction of the equivalence follows directly from the definition of a semi-topogenous order.

Now, to see that ϕ is open and continuous, note that for any point $x \in X$ and any open set U, we have

$$x \in U \Leftrightarrow \{x\} \sqsubset U \Leftrightarrow U \in \phi(x) \Leftrightarrow \phi(x) \in N_p,$$

where N_p is the basic open subset of $\mathrm{MF}(P)$ corresponding to U. $\square$

We do not know whether every MF space has a semi-topogenous order satisfying the hypotheses of the previous theorem. We have established the following partial result.

Theorem 7.5. If $X = \mathrm{MF}(P)$ and P satisfies

$$\forall p, q, r\,[p \prec q \wedge N_q \subseteq N_r \Rightarrow p \prec r] \tag{1}$$

then there is a semi-topogenous order $\sqsubset$ generating the topology of X such that X is complete for $\sqsubset$.

Proof. For any $v, w \subset X$, let $v \sqsubset w$ if either $v = \varnothing$, $w = X$, there is is an open atom u with $v \subseteq u \subseteq w$ or there are $p, q \in P$ with $v \subseteq N_p$, $p \prec q$ and $N_q \subseteq w$. Note that $N_p \subseteq N_q$ in the last case.

It follows directly from definitions and the present assumptions that $\sqsubset$ is a semi-topogenous order. We must show that that $\sqsubset$ generates the topology of X. Let w be an open set and x be a point in w. There is an open set N_q with $\{x\} \subseteq N_q \subseteq w$. In the case that $\{x\} = N_q$, $N_q \sqsubset w$. In the case that $\{x\} \neq N_q$ there is a further $p \prec q$ with $x \in N_p$: The reason is that given an $y \in N_q \setminus \{x\}$, the maximal filter U_x belonging to x must contain a $p \prec q$ which does not contain y by the T_1 axiom. Then $\{x\} \subseteq N_p \sqsubset w$. So w is the union of all v with $v \sqsubset w$.

Now let W be a filter in the topological space X which meets $\sqsubset$. If W contains an r such that N_r is atomic, that is, a singleton $\{x\}$, then every $u \in W$ contains x since otherwise $N_r \cap u = \varnothing$ in contradiction to W being a filter.

If W does not contain an r such that N_r is atomic, then let V be the set of all $p \in P$ such that $N_p \in W$. Given any $p, q \in V$ there is an u such that $u \sqsubset N_p \cap N_q$. So there are r, t with $u \subseteq N_r$, $N_t \subseteq N_p \cap N_q$ and $r \prec t$. It follows that $r \prec p$ and $r \prec q$. Thus V is the basis of a filter on P; this filter is contained in a maximal filter on P which is of the form U_x for some point $x \in X$. This x is then in N_p for all $p \in V$. Let $u \in W$. As W meets $\sqsubset$, there is a $p \in V$ with $N_p \subseteq u$. It follows that $x \in N_p$ and $x \in u$. So x is a common point of the sets in W. $\square$

The posets constructed in Theorems 2.2 and 2.3 satisfy condition (1) and thus they are examples of a poset space that is complete for a semi-topogenous order generating its topology.

Example 7.6. For every complete metric space and every locally compact Hausdorff space there exists a semi-topogenous order $\sqsubset$ which generates the topology of X and for which X is complete.

Remark 7.7. Assume that X is a space which is complete for a semi-topogenous order generating its topology. Then one can not only show that X is homeomorphic to an MF space but also that the winning strategy for player II is quite easy to obtain. Given any open set u and any point $x \in u$ by player I, player II only has to choose an open v with $\{x\} \subseteq v \sqsubset u$. It does not matter which v with this condition is chosen and the history of the game can be ignored. The result of the construction will be, at the end of the game, a basis for a filter which meets $\sqsubset$ and thus this filter has a common point.

This shows that the "neighbourhood spaces" that we consider here satisfy a restricted version of the strong Choquet property. The intuition behind this restriction is that one wishes to study non-second-countable spaces by considering "transfinite games." The role of player I is replaced by considering filters instead of descending sequences, and the winning strategy of player II is reduced to a neighbourhood relation $\sqsubset$ which could be interpreted as saying that if $\{x\} \subseteq v \sqsubset u$ then v is a good move for player II.

Indeed, the notion of completeness of spaces with respect to a semi-topogenous order $\sqsubset$ is based on this idea: Let the strategy of player II be just to follow $\sqsubset$ and let player I build a filter U such that for every $w \in U$ there is a $v \in U$ which player II might have chosen as a response to w, that is, a $v \sqsubset w$; then the intersection of all $u \in U$ is not empty.

8 Cardinality of poset spaces

In this section, we establish perfect set theorems for countably based Hausdorff poset spaces. These theorems show that these spaces are either countable or have the cardinality $2^{\aleph_0}$ of the continuum.

Theorem 8.1. Any countably based Hausdorff poset space has either countably many points or has cardinality $2^{\aleph_0}$.

Corollary 8.2. Any countably based Hausdorff poset space has either countably many points or contains a perfect closed set.

Proof. Any second-countable Hausdorff space of cardinality $2^{\aleph_0}$ contains a perfect closed set. The complement of the perfect closed set is the union of all the basic open sets from a fixed countable basis which contain fewer than $2^{\aleph_0}$ points. $\square$

To prove Theorem 8.1, we introduce a class of Gale–Stewart games. These games are inspired by the $*$-games in descriptive set theory (as described in [6]). For each poset P we define a game which we call the *poset star game on* P. There are two players. The play proceeds in stages numbered $0, 1, 2, \ldots$. At stage t, player I plays a pair $\langle p_1^t, p_2^t \rangle \in P \times P$. Player II plays a number $n_t \in \{1, 2\}$. Player I wins the game if the following conditions hold for all t.

(1) $p_1^t \perp p_2^t$.

(2) $p_1^{t+1} \preceq p_{n_t}^t$ and $p_2^{t+1} \preceq p_{n_t}^t$.

Player II wins if player I does not win; there are no ties.

A *strategy* for a player is a function that tells the player what to do at any possible move of the game. The strategy is a *winning strategy* if the player will win any play of the game in which the player uses the strategy to choose every move. It is impossible for both players to have a winning strategy for the same game.

Lemma 8.3. Let P be a poset. Either player I or player II has a winning strategy for the poset star game on P.

Proof. The set of infinite plays of the poset star game on P that are winning for player I is closed in the space of all possible plays of the game. (This space is the space of infinite sequences of moves; the set of moves is given the discrete topology and the space of infinite plays carries the product topology). The proof follows from a theorem of Gale and Stewart known as closed determinacy. $\square$

Lemma 8.4. Suppose that X is a Hausdorff poset space based on a countable poset P and player I has a winning strategy for the poset star game on P. Then X has cardinality $2^{\aleph_0}$.

Proof. It suffices to prove the result for $\mathrm{MF}(P)$, which is a subset of $\mathrm{UF}(P)$. Let s_{I} be a winning strategy for player I and let $f \in \{1, 2\}^{\mathbb{N}}$. Consider the play in which player I follows s_{I} while player II uses f as a guide; that

is, player II plays $f(n)$ at stage $2n$. Because s_{I} is a winning strategy for player I, this play determines a descending sequence $F(f)$ of elements of P. This sequence extends to a maximal filter. For distinct $f, g \in \{0,1\}^{\mathbb{N}}$ the sequences $F(f)$ and $F(g)$ contain incompatible elements and thus cannot extend to the same filter. Therefore the space $\mathrm{MF}(P)$ has cardinality $2^{\aleph_0}$. $\square$

Lemma 8.5. Let X be a countably based Hausdorff poset space based on the poset P. If player II has a winning strategy for the poset star game on P then X is countable.

Proof. Let s_{II} be a winning strategy for player II. We say that a finite play σ of length $2k$ is *compatible with* s_{II} if $s_{\mathrm{II}}(\sigma[2i+1]) = \sigma(2i+2)$ whenever $2i+2 \leq k$. We say that a play σ of even length is a *good play* for a point x if σ is compatible with s_{II} and x is in the open set chosen by player II in the last move of σ. A good play for x is a *maximal play* if it cannot be extended to a longer good play for x; this means that no matter what pair of disjoint open sets player I plays, s_{II} will direct player II to choose an open set not containing x.

If player II has a winning strategy then every point x has a maximal play. Note that the empty play is trivially a good play for x. If every good play for x could be extended to a larger good play for x, then it would be possible for player I to win the game by always leaving the game in a position that is good for x. This play of the game would follow s_{II}, a winning strategy for player II, which is a contradiction.

If σ is a good play for two points x and y then σ is not a maximal play for both x and y. For player I could play $\langle U_1, U_2\rangle$ in response to σ, where $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \varnothing$. Here we are using the assumption that the topology of X is Hausdorff.

We have now shown that every point in the X has a maximal play and that no play is maximal for two points. Since the set of maximal plays is countable, this implies that the set of points in X is countable. $\square$

We remark that the statement "Every closed subset of a countably based Hausdorff MF space is either countable or has a perfect closed subset" is independent of ZFC set theory; this result is established in [11].

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Carl Mummert
Department of Mathematics
Marshall University
One John Marshall Drive
Huntington, WV 25755
USA

Frank Stephan
School of Computing and Department of Mathematics
National University of Singapore
Singapore 117543
Republic of Singapore